MATH2050C Selected Solution to Assignment 7

Section 3.7 no. 3ac, 7, 10, 11, 12, 15, 16;

Section 4.1 no. 7, 8, 9bd, 10b, 11b, 12bd, 15;

Section 4.2 no. 1bc, 2bd.

Section 3.7

(11). Yes, $\sum a_n^2$ is convergent when $\sum a_n$ is convergent where $a_n \geq 0$. For, when the latter series is convergent, it implies in particular that $\{a_n\}$ is bounded. We can find some M such that $0 \leq a_n < M$. For $\varepsilon > 0$, there exists some n_0 such that $\sum_{k=m+1}^{n} a_n < \varepsilon/M$ for all $n, m \geq n_0$. But then

$$\sum_{k=m+1}^{n} a_k^2 \leq M \sum_{k=m+1}^{n} a_k < M\frac{\varepsilon}{M} = \varepsilon \ ,$$

so $\sum a_n^2$ is convergent by Cauchy Convergence Criterion.

(12). No. It suffices to consider $\sum 1/n^2$.

(15). Use induction to show

$$\frac{1}{2}(a(1) + 2a(2) + \cdots + 2^n a(2^n)) \leq s(2^n) \leq (a(1) + 2a(2) + \cdots + 2^{n-1}a(2^{n-1})) + a(2^n) \ ,$$

where $a_n > 0$ is decreasing. We work out the right inequality and leave the left one to you. When $n = 1$, the right inequality becomes

$$a(1) + a(2) \leq a(1) + a(2),$$

which is trivial. Assume it is true for n and we establish it for $n+1$. Indeed, by induction hypothesis and the fact that $\{a_n\}$ is decreasing,

$$\begin{aligned}
s(2^{n+1}) &= a(1) + a(2) + \cdots + a(2^n) + a(2^n + 1) + \cdots + a(2^{n+1}) \\
&= s(2^n) + a(2^n + 1) + \cdots + a(2^{n+1}) \\
&\leq \big(a(1) + \cdots + 2^{n-1}a(2^{n-1}) + a(2^n)\big) + a(2^n + 1) + \cdots + a(2^{n+1}) \\
&= a(1) + \cdots + 2^{n-1}a(2^{n-1}) + \big(a(2^n) + a(2^n + 1) + \cdots + a(2^{n+1} - 1)\big) + a(2^{n+1}) \\
&\leq a(1) + \cdots + 2^{n-1}a(2^{n-1}) + 2^n a(2^n) + a(2^{n+1}) \ ,
\end{aligned}$$

done.

(16). We look at $\sum_{n=1}^{\infty} 2^n a(2^n) = \sum_{n=1}^{\infty} 2^n/2^{np} = \sum_{n=1}^{\infty} 2^{(1-p)n}$, which is convergent if and only if $p > 1$. We conclude that the p-series is convergent if and only if $p > 1$.

Section 4.1

(9d). We use ε-δ definition. Consider

$$\left|\frac{x^2 - x + 1}{x + 1} - \frac{1}{2}\right| = \left|\frac{2x^2 - 3x + 1}{2(x + 1)}\right| = \left|\frac{2x - 1}{x + 1}\right| |x - 1|.$$

We make a first choice $\delta_1 = 1/2$. Then for $|x - 1| < 1/2$, that is, $1/2 < x < 3/2$. Then $|2x - 1|/|x + 1| \leq 4/3$. Therefore, for $\delta = \min\{\delta_1, 3\varepsilon/4\}$, we have

$$\left|\frac{x^2 - x + 1}{(x + 1) - 1/2}\right| < \frac{4}{3}|x - 1| < \varepsilon \ ,$$

for $x, 0 < |x-1| < \delta$.

Or, we could use Sequential Criterion. Let $\lim_{n\to\infty} x_n = 1$. By Limit Theorem $\lim_{n\to\infty}(x_n^2 - x_n + 1) = 1$ and $\lim_{n\to\infty}(x_n+1) = 2$. Therefore,

$$\lim_{n\to\infty} \frac{x_n^2 - x_n + 1}{x_n + 1} = \frac{\lim_{n\to\infty}(x_n^2 - x_n + 1)}{\lim_{x\to\infty}(x_n+1)} = \frac{1}{2} \ .$$

(12d). We claim that $\lim_{x\to 0} \sin(1/x^2)$ does not exist. Take the sequence $x_n = \sqrt{1/(2n\pi)}$ and $y_n = \sqrt{1/(2n\pi + \pi/2)}, n \geq 1$. Both sequences tend to 0 as $n \to \infty$. As $\lim_{n\to\infty} \sin(1/x_n^2) = 0$ and $\lim_{n\to\infty} \sin(1/y_n^2) = 1$, they have different limit. We conclude that the limit of $\sin(1/x^2)$ as $x \to 0$ does not exist.

(15). (a) We want to show $\lim_{x\to 0} f(x) = 0$ where f is the function that is equal to x at rational x and 0 at irrational x. The desired conclusion follows from the observation $|f(x)| \leq |x|$ and $\lim_{x\to 0} |x| = 0$ and the Squeeze Theorem.

(b) f has no limit at $x = c \neq 0$. Let $x_n \to c$ be a sequence of rational numbers. Clearly, $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = c$. But, take $y_n \to c$ be a sequence of irrational numbers, then $f(y_n) = 0$, so $\lim_{n\to\infty} f(y_n) = 0$. From Sequential Criterion we draw the desired conclusion.

Section 4.2

(1b). Since the limit is taken among positive x only, this should be viewed as a right limit (see below). By Limit Theorem,

$$\lim_{x\to 1^+} \frac{x^2+2}{x^2-2} = \frac{\lim_{x\to 1^+}(x^2+2)}{\lim_{x\to 1^+}(x^2-2)} = \frac{3}{-1} = -3.$$

Supplementary Problems

1. An infinite series $\sum_n x_n$ is called **absolutely convergent** if $\sum_n |x_n|$ is convergent. Show that an absolutely convergent infinite series is convergent but the convergence of $\sum_n x_n$ does not necessarily imply the convergence of $\sum_n |x_n|$.

   Solution. By Cauchy Convergence Criterion, when $\sum |x_n|$ is convergent, for each $\varepsilon > 0$, there is some n_0 such that

   $$\sum_{k=m+1}^{n} |x_k| < \varepsilon, \quad \forall n, m \geq n_0.$$

   But then by the triangle inequality it implies

   $$\left| \sum_{k=m+1}^{n} x_k \right| \leq \sum_{k=m+1}^{n} |x_k| < \varepsilon, \quad \forall n, m \geq n_0 \ ,$$

   in other words, the sequence of partial sums for $\sum x_n$ forms a Cauchy sequence and hence is convergent.

   The series $\sum_{n=1}^{\infty} (-1)^{n+1}/n$ is convergent but $\sum_{n=1}^{\infty} 1/n$ is divergent.

2. Suppose that for a polynomial p and $c \in \mathbb{R}$, prove by the Limit Theorem (see next page) that $\lim_{x\to c} p(x) = p(c)$.

Solution. Using $\lim_{x\to c} x = c$ for all c, the product rule tells us that $\lim_{x\to c} x^k = (\lim_{x\to c} x)(\lim_{x\to c} x)\cdots(\lim_{x\to c} x) = c^k$ (k-times product). Hence

$$\begin{aligned}\lim_{x\to c} p(x) &= \lim_{x\to c}(a_0 + a_1 x + \cdots + a_n x^n) \\ &= a_0 \lim_{x\to c} 1 + a_1 \lim_{x\to c} x + \cdots + a_n \lim_{x\to c} x^n \\ &= a_0 + a_1 c + \cdots + a_n c^n \\ &= p(c)\ .\end{aligned}$$

3. Let f be defined on (a,b) possibly except $x_0 \in (a,b)$. Show that $\lim_{x\to x_0} |f(x)| = |L|$ whenever $\lim_{x\to x_0} f(x) = L$.

   Solution. It follows immediately from the triangle inequality $||f(x)| - |L|| \le |f(x) - L|$.

4. Let f be defined on (a,b) possibly except $x_0 \in (a,b)$. Suppose that $\lim_{x\to x_0} f(x) = L$ for some L. Show that $\lim_{x\to x_0} \sqrt{f(x)} = \sqrt{L}$ provided $f \ge 0$ on (a,b). Suggestion: Consider $L > 0$ and $L = 0$ separately.

   Solution. First, assume $L > 0$. Given $\varepsilon = L/2 > 0$, there is some δ_1 such that $|f(x) - L| \le L/2$ for $0 < |x - x_0| < \delta_1$. In particular, it implies that $f(x) \ge L/2$ for $0 < |x - x_0| < \delta_1$. Now,

   $$|\sqrt{f(x)} - L^{1/2}| = \frac{|f(x) - L|}{\sqrt{f(x)} + L^{1/2}} \le \frac{1}{(L/2)^{1/2} + L^{1/2}} \times |f(x) - L|\ ,$$

   for $0 < |x - x_0| < \delta_1$. For $\varepsilon > 0$, there is δ_2 such that $|f(x) - L| < \varepsilon \times [(L/2)^{1/2} + L^{1/2}]$ for $x, 0 < |x - x_0| < \delta_2$. If we take $\delta = \min\{\delta_1, \delta_2\}$, then

   $$|\sqrt{f(x)} - L^{1/2}| < \frac{1}{(L/2)^{1/2} + L^{1/2}} \times |f(x) - L| < \varepsilon\ , \quad \forall x, 0 < |x - x_0| < \delta,$$

   done.

   Next, $L = 0$. Given $\varepsilon > 0$, there is some δ such that $|f(x)| < \varepsilon^2$ for all $x, 0 < |x - x_0| < \delta$. It follows that $|\sqrt{f(x)} - 0| = \sqrt{f(x)} < \varepsilon$ for $x, 0 < |x - x_0| < \delta$, done.